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New Geometrical Approach to Rainich–Misner–Wheeler Theory

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We introduce a new principal fiber bundle, the bundle of biframes, associated with the geometry of bivectors on spacetime. It is shown that the biframe bundle is a natural geometric arena for modeling the already unified theory of Rainich, Misner, and Wheeler (RMW). The structure equations for the bitorsion inherent in the biframe bundle lead to a generalization of Rainich's algebraic conditions for electromagnetic-type stress tensors which includes sources in a natural way. Besides the usual complexion vector of the RMW theory, an additional new complexion-type vector is found. The generalized algebraic conditions reduce to the usual RMW conditions in the special case of no sources.

1. INTRODUCTION

The already unified field theory of Rainich (1925) and Misner and Wheeler (1957) (RMW) is a geometrically unified theory of gravity and source-free electromagnetism. The standard geometrical arena for the RMW problem is four-dimensional Riemannian geometry, and the RMW theory provides the necessary and sufficient conditions for a spacetime (M, g) to be a source-free Einstein–Maxwell spacetime. Within this arena, however, there has never been a completely geometrical formulation nor a satisfactory method to extend the RMW program to include Einstein–Maxwell spacetimes with sources. In particular, sources for the electromagnetic field and nonelectromagnetic sources for the gravitational field have been treated in neither a geometrical nor a nongeometrical manner within the framework of the standard RMW theory.

In this paper we show that the RMW problem has a natural formulation in a new geometrical arena, namely the biframe bundle associated with the

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spacetime manifold M. The geometrical richness of the biframe bundle provides a natural geometrical setting for the original RMW problem and allows a generalization of the original RMW theory to include geometrical sources for the Maxwell equations. Furthermore, it will be shown that the RMW formalism can be extended to include nongeometrical gravitational sources. The generalization introduces a second complexion vector in addition to the usual RMW complexion vector and reduces, in the special case of no sources, to the standard RMW formalism.

Riemannian geometry on a manifold M is usually considered in terms of a metric tensor on M and the associated Levi-Civita linear connection on the bundle of linear frames LM. Indeed, the geometry of linear connections has LM as the fundamental arena. This standard picture of geometry is based on the fact that all higher order tensors on M can be built up from tensor products of vectors and covectors, and the assumption that the transformation properties of such tensors are only induced by changes of linear frames. However, the tensor geometry of a manifold is much richer in the sense that there exist tensor frame bundles with linear connections that cannot be induced from connections on LM.

Thus, for example, if one considers a tensor field T of type (r, s) on the spacetime manifold M, then at each $p \in M$ the tensor $T(p)$ is an element of the tensor space $T^r_sM_p$ which is itself a vector space of dimension $N = (\dim M)^{r+s}$. Furthermore, linear frames (bases) for each of these vector spaces $T^r_sM_p$ can be used to construct a corresponding tensor frame bundle L^r_sM. This bundle consists of pairs $(p, (\pi_A))$, $A = 1, \ldots, N$, where (π_A) is a linear frame for $T^r_sM_p$.

As an explicit example of a tensor frame bundle, consider tensors of type $(1, 2)$ on a four-dimensional spacetime manifold M. Let (e_α) be a linear frame for T^1M_p. As is well known, the vector space T^1M_p is a four-dimensional space with transformation group $Gl(4)$; that is, frames for T^1M_p transform under $Gl(4)$.

In a similar manner, the space $T^1_2M_p$ forms a vector space that is sixty-four-dimensional, with corresponding structure group $Gl(64)$. Let (π_A) be a basis of this vector space $(A = 1, \ldots, 64)$ with dual basis (π^B), $\pi^B(\pi_A) = \delta^B_A$. Since elements of $T^1_2M_p$ can be built up from tensor products of elements of T^1M_p and T_1M_p, this basis can be written as $\pi_A = \pi^{\alpha\beta}_{A\gamma}(\mathscr{E}^\gamma \otimes e_\alpha \otimes e_\beta)$, where $(\mathscr{E}^\gamma)$ is dual to (e_α). L^1_2M, the set of all frames for $T^1_2M_p$ at all points of M, can be given a manifold structure such that $L^1_2M \xrightarrow{\pi} M$ is a principal fiber bundle with structure group $Gl(64)$.

The above relations between the basis of T^1M_p and $T^1_2M_p$ shows clearly that $Gl(4)$ transformations on the basis (e_α) can induce some transformations on the (π_A) basis. However, the (π_A) will transform in general under the full $Gl(64)$ group. Clearly, not all $Gl(64)$ transformations can be induced by $Gl(4)$ transformations. In this sense the tensor frame bundle is much richer in structure than just that induced by the frame bundle LM.

In a similar manner, a linear connection on LM can induce a connection on L_2^1M. For simplicity, assume a Riemannian connection on LM. The pullback of the Ricci identities from LM to M for a tensor of type $(1,2)$ can be written in local coordinates as

$$\nabla_{[\alpha}\nabla_{\beta]}T^{\rho}_{\gamma\delta}=\tfrac{1}{2}(-R^{\sigma}_{\alpha\beta\gamma}T^{\rho}_{\sigma\delta}-R^{\sigma}_{\alpha\beta\delta}T^{\rho}_{\gamma\sigma}+R^{\rho}_{\alpha\beta\mu}T^{\mu}_{\gamma\delta}) \tag{1.1}$$

where ∇_α is the local covariant derivative operator with respect to the Riemannian connection and $R^{\sigma}_{\alpha\beta\delta}$ are the components of the Riemannian (LM) curvature tensor. This notation follows Schouten (1954) and will be used throughout. These identities can be rewritten as

$$\nabla_{[\alpha}\nabla_{\beta]}T^{\rho}_{\gamma\delta}=\tfrac{1}{2}\mathring{R}^{\rho\theta\phi}_{\alpha\beta\gamma\delta\sigma}T^{\sigma}_{\theta\phi} \tag{1.2}$$

where

$$\mathring{R}^{\rho\theta\phi}_{\alpha\beta\gamma\delta\sigma}=(-R^{\theta}_{\alpha\beta\gamma}\delta^{\phi}_{\delta}\delta^{\rho}_{\sigma}-R^{\phi}_{\alpha\beta\delta}\delta^{\theta}_{\gamma}\delta^{\rho}_{\sigma}+R^{\rho}_{\alpha\beta\sigma}\delta^{\theta}_{\gamma}\delta^{\phi}_{\delta}) \tag{1.3}$$

Here, $\mathring{R}$ can be thought of as the coordinate form of a curvature tensor on L_2^1M that is induced from the frame bundle curvature tensor. However, such a curvature tensor is only a special case of a general curvature on L_2^1M.

Connections on L_2^1M can be defined in the standard way analogous to the definition of connections on LM, i.e., by $gl(64)$-valued one-forms with standard transformation properties [see, for example, Kobayashi and Nomizu (1963)]. Given such a connection on L_2^1M, the components of the pullback to M can be denoted as $\Gamma^{A}_{\mu B}$ ($A, B=1,\ldots,64$). The local coordinate expression of the corresponding $gl(64)$-valued curvature two-form can then be written as $R^{B}_{\alpha\beta A}$ [cf. equation (2.8)]. A related object can be defined as

$$R^{\rho\theta\phi}_{\alpha\beta\gamma\delta\sigma}=R^{B}_{\alpha\beta A}\pi^{A\rho}_{\gamma\delta}\pi^{\theta\phi}_{B\sigma} \tag{1.4}$$

The curvature (1.4) will not, in general, be just an induced curvature of the type given in (1.3), and in fact a general connection on L_2^1M is independent of connections on LM. On the other hand, given a connection on LM, one can always induce a special connection on L_2^1M as discussed above.

In this paper we will be concerned with only one of these tensor bundles, namely the principal bundle of biframes. The tensor spaces associated with this frame bundle will be AT^2M_p, i.e., antisymmetric rank-two tensors of type $(2,0)$ (*bivectors*) at each $p\in M$. A basis of AT^2M_p will be called a *biframe*, and we will denote the biframe bundle by L^2M. For a four-dimensional spacetime manifold M, each AT^2M_p is six-dimensional. Thus, a biframe consists of six independent bivectors at a spacetime point. A basis of the dual space, Λ^2M_p, with cobiframe bundle L^2M^*, consists of six independent *cobivectors* (two-forms) at a spacetime point p.

The cobiframe bundle can be considered physically interesting for a number of reasons. In particular, the cobiframe bundle is a natural geometrical arena for gauge theories. The intuitive idea is the following. Suppose one is given a set of independent gauge curvature two-forms (Yang-Mills fields) on the spacetime manifold. If the number of gauge curvatures is less than or equal to six, these two-forms can be used to construct all or part of a basis of $\Lambda^2 M_p$. Then geometric structures on L^2M^* can be expressed fundamentally in terms of these special cobiframes.

One of the main features of the RMW problem is the existence of naturally defined geometrical bivectors, the extremal Maxwell square root of the Ricci tensor and its dual. Norris and Davis (1979) have classified source-free Einstein-Maxwell spacetimes in terms of the natural Riemannian bivector structure, i.e., in terms of the infinitesimal holonomy group (IHG). The approach uses a classification scheme due to Schell (1961). In particular, they have shown that the extremal fields (and thus by a duality rotation the Maxwell fields) can always be identified with two of the generators of the IHG. As we will see, a key step in modeling the RMW problem on the biframe bundle will be to pick the extremal fields as part of the bivector basis.

2. THE BIFRAME BUNDLE

Before considering a sketch of the structure of the biframe bundle (a more detailed account will be published elsewhere) a sketch of the standard frame bundle LM will be given. Assume a four-dimensional spacetime manifold M. The frame bundle LM is a principal fiber bundle with structure group $Gl(4)$. A point $u \in LM$ can be written as $u = (p, e_\alpha)$, where (e_α) is a basis of T^1M_p, with dual basis $(\mathscr{E}^\alpha)$ $(\mathscr{E}^\alpha(e_\beta) = \delta^\alpha_\beta,\ \alpha, \beta = 1, \ldots, 4)$. The projection π: $LM \to M$ is defined by $\pi(u) = p$. A local section (tetrad field) s: $U \to LM$, $U \subseteq M$, can be defined as $s(p) = (p, e_\alpha|_p)$.

The frame bundle LM is unique among $Gl(4)$ principal bundles over spacetime in that it supports an object called the soldering form θ [see, for example, Trautman (1970) and Norris *et al.* (1980)]. The soldering form on LM is an $\mathbb{R}^4$-valued one-form, i.e., θ: $T_uLM \to \mathbb{R}^4$ and is defined by $\theta_u(X) = \mathscr{E}^\alpha[d\pi(X)]r_\alpha$, where (r_α) is the standard basis of $\mathbb{R}^4$, $u = (p, e_\alpha)$, and $X \in T_uLM$. The soldering form on LM is characterized by the following properties:

(a) θ is an $\mathbb{R}^4$-valued one-form on LM.
(b) $R_g^*\theta = g^{-1} \cdot \theta, \forall g \in Gl(4)$.
(c) $\theta(X) = 0$ if and only if $d\pi(X) = 0$.

In (b) the "dot" denotes the standard action of $Gl(4)$ on $\mathbb{R}^4$.

Given a connection $\tilde{\omega}$ on LM, the curvature and torsion two-forms are defined by (Kobayashi and Nomizu, 1963)

$$\tilde{\Omega} = \tilde{D}\tilde{\omega} = d\tilde{\omega} + \tilde{\omega} \wedge \tilde{\omega} \tag{2.1}$$

$$\tilde{\Theta} = \tilde{D}\theta = d\theta + \tilde{\omega} \wedge \theta \tag{2.2}$$

where $\tilde{D}$ denotes the exterior covariant derivative with respect to $\tilde{\omega}$. The torsion $\tilde{\Theta}$ is an $\mathbb{R}^4$-valued two-form. Pulled back in a local gauge $s(p) = (p, e^i_\alpha \partial_i|_p)$ of LM, (2.2) takes the form

$$_s\tilde{\Theta}^\alpha_{jk} = 2(\partial_{[j}\mathscr{E}^\alpha_{k]} + \Gamma^\alpha_{[j|\beta|}\mathscr{E}^\beta_{k]}) \tag{2.3}$$

The biframe bundle L^2M is defined over the same four-dimensional spacetime manifold M. The associated tensor spaces are the spaces AT^2M_p of antisymmetric rank-two tensors at $p \in M$. As a vector space, AT^2M_p is six-dimensional. Let (t_a) $(a = 1, \ldots, 6)$ be a basis (a biframe) of AT^2M_p. The dual basis is denoted by (τ^a), and satisfies $\tau^a(t_b) = \delta^a_b$. Since the t_a are antisymmetric rank-two tensors (bivectors), they can be written as

$$t_a = \tfrac{1}{2} t_a^{\alpha\beta}(e_\alpha \otimes e_\beta - e_\beta \otimes e_\alpha) \overset{d}{=} t_a^{\alpha\beta}(e_\alpha \wedge e_\beta), \qquad \alpha < \beta$$

for (e_α) a basis of T^1M. Further, the t_a transform, in general, under $Gl(6)$. That is, if $g \in Gl(6)$, $g = (g^a_b)$ $(a, b = 1, \ldots, 6)$, then a new biframe basis $\bar{t}_a$ can be defined by $\bar{t}_a = t_b g^b_a$. Since there are six independent bivectors in the biframe, one needs the full $Gl(6)$ transformation group.

The biframe bundle L^2M is a principal fiber bundle with structure group $Gl(6)$. A point $u \in L^2M$ can be written as $u = (p, t_a)$, where $(t_a)_p$ is a biframe at $p \in M$. The projection π: $L^2M \to M$ is defiend by $\pi(u) = p$. A local section σ: $U \to L^2M$, $U \subseteq M$, is defined by $\sigma(p) = (p, t_a|_p)$, for all points $p \in U$. The right action of $Gl(6)$ on L^2M is defined by $R_g : L^2M \to L^2M$ such that $R_g u = u \cdot g = (p, t_a g^a_b)$, $\forall g \in Gl(6)$. A connection 1-form on L^2M is a $gl(6)$-valued 1-form with the standard properties of a connection.

The biframe bundle does support a generalized soldering form. However, the striking difference is that this soldering form is a *two-form*, as opposed to the one-form on LM. Let β be an $\mathbb{R}^6$-valued two-form on L^2M, i.e., β_u: $T_uL^2M \times T_uL^2M \to \mathbb{R}^6$, defined by $\beta_u(X, Y) = \tau^a(d\pi(X), d\pi(Y))r_a$ for $X, Y \in T_uL^2M$, $u = (p, t_a)$, and (τ^a) dual to (t_a). Here and in the following (r_a) denotes the standard basis of $\mathbb{R}^6$.

The soldering form on L^2M has the following properties:

(a) β is an $\mathbb{R}^6$-valued two-form on L^2M.
(b) $R^*_g\beta = g^{-1} \cdot \beta$, $\forall g \in Gl(6)$.
(c) $\beta(X, Y) = 0$ if $d\pi(X) = 0$ and/or $d\pi(Y) = 0$.

In (b) the "dot" denotes the standard action of $Gl(6)$ on $\mathbb{R}^6$.

Given a connection one-form ω on L^2M, the curvature of ω is the $gl(6)$-valued two-form defined in the standard way by

$$\Omega = D\omega = d\omega + \omega \wedge \omega \tag{2.4}$$

However, the bitorsion of the connection is an $\mathbb{R}^6$-valued tensorial *three-form*, defined by

$$\Theta = D\beta = d\beta + \omega \wedge \beta \tag{2.5}$$

The bitorsion and the curvature on L^2M satisfy the first and second Bianchi identities

$$D\Theta = \Omega \wedge \beta \tag{2.6}$$

$$D\Omega = 0 \tag{2.7}$$

Let $\sigma\colon U \to L^2M$, $U \subseteq M$, be a local section of L^2M with $\sigma(p) = (p, t_a|_p)$, $\forall p \in U$. The components of the pullback of the curvature take the standard form (where $\Gamma = \sigma^*\omega$, $R = \frac{1}{2}\sigma^*\Omega$)

$$R^{\ b}_{\alpha\beta a} = 2(\partial_{[\alpha}\Gamma^{\ b}_{\beta]a} + \Gamma^{\ b}_{[\alpha|c|}\Gamma^{\ c}_{\beta]a}) \tag{2.8}$$

Associated objects can be defined by using a biframe (t_a) and its dual (τ^a) to express Lie algebra indices as spacetime indices. For example, $R^{\ b}_{\alpha\beta a}$ can be reexpressed as $R^{\ \sigma\rho}_{\alpha\beta\gamma\delta} = R^{\ b}_{\alpha\beta a}\tau^{a}_{\gamma\delta}t^{\sigma\rho}_{b}$. The local components of the pullback of the bitorsion (2.5) and the first Bianchi identity (the bitorsion Bianchi identity) (2.6) take the forms

$${}_\sigma\Theta^{\ a}_{\alpha\beta\gamma} = \partial_{[\alpha}\tau^{a}_{\beta\gamma]} + \Gamma^{\ a}_{[\alpha|b|}\tau^{b}_{\beta\gamma]} \tag{2.9}$$

$$\partial_{[\alpha}({}_\sigma\Theta)^{a}_{\beta\gamma\delta]} + \Gamma^{\ a}_{[\alpha|b|}({}_\sigma\Theta)^{b}_{\beta\gamma\delta]} = \tfrac{1}{2}R^{\ a}_{[\alpha\beta|b|}\tau^{b}_{\gamma\delta]} \tag{2.10}$$

respectively, where ${}_\sigma\Theta = \sigma^*\Theta$. Here and in the following we use a left subscript to denote the gauge when needed for clarity.

3. THE BITORSION STRUCTURE EQUATIONS AND GEOMETRICAL SOURCES

As a simple preliminary application of the formalism, we show that the bitorsion equations (2.9) and (2.10) can be rewritten in a form which is very analogous to the field equations that occur in non-Abelian gauge theories. Consider a typical non-Abelian gauge theory based on some N-parameter group. Let A^a_μ ($a = 1, \ldots, N$) represent the local components of a given connection 1-form (the gauge potentials) with values in the Lie algebra of the gauge group. Further, let $F^a_{\mu\nu}$ be the components of the corresponding curvature two-forms (the field strengths). Then the field equations of such a typical gauge theory can be written as

$$\nabla_\alpha F^{*\mu\alpha a} = -A^a_{\alpha b} F^{*\mu\alpha b} + J^{\mu a}_M$$
$$= J^{\mu a}_{SM} + J^{\mu a}_M = J^{\mu a}_{M,\text{total}} \tag{3.1}$$
$$\nabla_\alpha F^{\mu\alpha a} = -A^a_{\alpha b} F^{\mu a b} + J^{\mu a}_E$$
$$= J^{\mu a}_{SE} + J^{\mu a}_E = J^{\mu a}_{E,\text{total}} \tag{3.2}$$

where we have defined $A^a_{\mu b} = {}^{d}\tfrac{1}{2} f^a_{cb} A^c_\mu$, where the f^a_{cb} are the structure constants of the Lie algebra. In the above, J_E and J_M are the generalized external electric and magnetic sources, while J_{SE} and J_{SM} are the electric and magnetic self-currents. The external sources transform tensorially with respect to a transformation of the gauge group, while the self-currents transform in a nontensorial manner. Conservation of the total currents, namely $\nabla_\mu J^{\mu a}_{M,\text{total}} = 0$ and $\nabla_\mu J^{\mu a}_{E,\text{total}} = 0$, follows as a trivial consequence of the structure of the field equations.

Consider the bitorsion structure equations (2.9) and the bitorsion Bianchi identities (2.10). First, since ${}_\sigma\Theta$ is a three-form, we can use the Hodge dual operator defined by the metric to define equivalent one-forms ${}^*_\sigma\Theta^{\lambda a} = \tfrac{1}{2}\mathscr{E}^{\lambda\alpha\beta\gamma}({}_\sigma\Theta)^a_{\alpha\beta\gamma}$. Next, since the bitorsion Bianchi identity is expressed in terms of a four-form, we can take the total dual of (2.10) to obtain an equivalent scalar equation. Further, if a Riemannian connection on LM is assumed, then relations (2.9) and (2.10) can be recast in the form

$$\nabla_\alpha \tau^{*\mu\alpha a} = {}_\sigma J^{\mu a} + {}^*_\sigma\Theta^{\mu a} \overset{d}{=} {}_\sigma J^{\mu a}_{\text{total}} \tag{3.3}$$

and

$$\nabla_\alpha({}_\sigma J^{\alpha a} + {}^*_\sigma\Theta^{\alpha a}) = \nabla_\alpha({}_\sigma J^{\alpha a}_{\text{total}}) = 0 \tag{3.4}$$

respectively. In the above relations ∇_α is the local covariant derivative operator with respect to the Riemannian connection on LM. Further, we have defined a biframe "self-current" as ${}_\sigma J^{\mu a} = -{}_\sigma\Gamma^a_{\mu b}\tau^{*\mu\alpha b}$.

This bitorsion structure equations (3.3) are completely analogous to the gauge theory field equations (3.1) and (3.2). The bitorsion transforms tensorially under a $Gl(6)$ transformation, while the biframe self-current does not. In this analogy *the bitorsion plays the role of an external geometrical source* in the bitorsion structure equations. Furthermore, *the bitorsion Bianchi identities* (3.4) *guarantee conservation of the total geometrical sources.* The current conservation laws are a consequence of the biframe bundle geometry. These results will be used below in reformulating the RMW problem.

4. THE STANDARD RMW THEORY

We next recall some basic facts from RMW theory (Rainich, 1925; Misner and Wheeler, 1957). A source-free Einstein–Maxwell spacetime is any four-dimensional Riemannian spacetime which satisfies

$$\tilde{G}_{\mu\nu} = f_{\alpha\mu} f^{\alpha}_{\nu} + f^{*}_{\alpha\mu} f^{*\alpha}_{\mu} \tag{4.1}$$

$$\nabla_{\alpha} f^{*\lambda\alpha} = 0 \tag{4.2}$$

$$\nabla_{\alpha} f^{\lambda\alpha} = 0 \tag{4.3}$$

where $\tilde{G}_{\mu\nu}$ is the Einstein tensor and $f_{\mu\nu}$ is the Maxwell field strength. Further, a non-null field satisfies $f^{*}_{\theta\phi} f^{\theta\phi} \neq 0$.

The RMW theory provides a method of geometrizing such spacetimes. The following conditions are both necessary and sufficient for an arbitrary four-dimensional Riemannian spacetime (M, g) to be equivalent to a non-null source-free Einstein–Maxwell spacetime:

$$\tilde{R} = 0 \tag{4.4}$$

$$\tilde{R}^{\beta}_{\alpha} \tilde{R}^{\gamma}_{\beta} = \tfrac{1}{4} \delta^{\gamma}_{\alpha} \tilde{R}_{\theta\phi} \tilde{R}^{\theta\phi} \tag{4.5}$$

$$\tilde{R}_{00} \geq 0 \tag{4.6}$$

$$\alpha_{\mu} = \partial_{\mu} \alpha \tag{4.7}$$

where

$$\alpha_{\mu} = \mathscr{E}_{\mu\lambda\gamma\nu} \left\{ \frac{\tilde{R}^{\nu}_{\sigma} \nabla^{\gamma} \tilde{R}^{\lambda\alpha}}{\tilde{R}_{\theta\phi} \tilde{R}^{\theta\phi}} \right\} \tag{4.8}$$

Non-null Einstein–Maxwell spacetimes are geometrized in that conditions (4.4)–(4.8) are purely geometrical relations stated completely in terms of g and its derivatives. In particular, the nongeometrical Maxwell field strength $f_{\mu\nu}$ does not explicitly appear in these conditions. This nonappearance of $f_{\mu\nu}$ in equations (4.4)–(4.8) is a strength of the RMW theory in that the conditions can be stated completely in terms of the metric. On the other hand, the physical Maxwell field $f_{\mu\nu}$ does *not* play a fundamental geometrical role in the theory.

Any four-dimensional Riemannian geometry which satisfies (4.4)–(4.6) will be called a non-null *algebraic RMW spacetime* (*ARMW*), while (4.7)–(4.8) will be referred to as the RMW *differential condition*. The vector α_{μ} in (4.8) is called the *complexion vector*.

The electromagnetic field strength can be recovered in the RMW theory. Given an algebraic RMW spacetime, there exists naturally defined geometrical bivectors $\xi_{\alpha\beta}$ and its dual $\xi^{*}_{\alpha\beta}$. The bivector $\xi_{\alpha\beta}$ is the so-called *extremal Maxwell square root* of the Ricci tensor (Rainich, 1925; Misner and Wheeler, 1957).

For ARMW spacetimes, new bivectors $\Sigma_{\alpha\beta}$ and $\Sigma^*_{\alpha\beta}$ can be constructed by a *duality rotation*, i.e.,

$$\Sigma_{\alpha\beta} = \xi_{\alpha\beta} \cos \alpha + \xi^*_{\alpha\beta} \sin \alpha \tag{4.9}$$

$$\Sigma^*_{\alpha\beta} = \xi^*_{\alpha\beta} \cos \alpha - \xi_{\alpha\beta} \sin \alpha \tag{4.10}$$

The algebraic conditions (4.4)-(4.6) are necessary and sufficient to guarantee that the new bivectors $\Sigma_{\alpha\beta}$ and $\Sigma^*_{\alpha\beta}$, for each complexion angle α, satisfy the quadratic form

$$\begin{aligned}\tilde{G}_{\mu\nu} &= \xi_{\alpha\mu}\xi^{\alpha}_{\nu} + \xi^*_{\alpha\mu}\xi^{*\alpha}_{\nu} \\ &= \Sigma_{\alpha\mu}\Sigma^{\alpha}_{\nu} + \Sigma^*_{\alpha\mu}\Sigma^{*\alpha}_{\nu}\end{aligned} \tag{4.11}$$

as in (4.1).

Thus, ARMW spacetimes guarantee the quadratic structure (4.1), but the Maxwell equations need not be satisfied. The extra condition necessary and sufficient for the source-free Maxwell equations to be satisfied is precisely the RMW differential condition (4.7)-(4.8).

Recall that when the differential condition (4.7)-(4.8) is satisfied, i.e., when $\alpha_\mu = \partial_\mu \alpha$, the Maxwell equations (4.2) and (4.3) can be written in terms of the extremal Maxwell square root as

$$\nabla_\mu \xi^{*\lambda\mu} - (\partial_\mu \alpha)\xi^{\lambda\mu} = 0 \tag{4.12}$$

$$\nabla_\mu \xi^{\lambda\mu} + (\partial_\mu \alpha)\xi^{*\lambda\mu} = 0 \tag{4.13}$$

The relations (4.12) and (4.13) will be called the *RMW extremal field equations.* Thus, (4.12) and (4.13) are equivalent to the Maxwell equations in that a duality rotation on the bivectors occurring in (4.12) and (4.13) will produce the source-free Maxwell equations (4.2) and (4.3).

5. THE BIFRAME BUNDLE AS A NATURAL GEOMETRICAL ARENA FOR THE RMW THEORY

The biframe bundle is a natural geometrical arena for the RMW problem. The intuitive idea of the construction described below is as follows. If one is given an ARMW spacetime, then, as discussed earlier, there exist naturally defined geometrical bivectors $\xi_{\alpha\beta}$ and $\xi^*_{\alpha\beta}$. As will be shown below, these bivectors can be used to define special sections of L^2M. Further, it will be shown that duality rotations correspond precisely to special section changes on the biframe bundle.

However, the real power of studying ARMW spacetimes on the biframe bundle lies in the bitorsion structure equations. We will show that when equations (3.3) are pulled back in one of the above-mentioned special sections they take the form of generalized extremal equations. These geometrical bivector equations, inherent to the biframe bundle, reduce in a special case to the RMW extremal field equations (4.12) and (4.13).

Thus, to model the RMW problem on the biframe bundle, consider the set of all ARMW spacetimes. Each such ARMW spacetime leads to the geometrical bivectors $\xi_{\alpha\beta}$ and $\xi^*_{\alpha\beta}$. An equivalence class $[{}^*\xi]$ of sections of L^2M^* can then be defined. Two sections ${}^*\xi$ and ${}^*\eta \in [{}^*\xi]$ are equivalent if ${}^*\xi(p) = (p, (\xi_{\alpha\beta}, -\xi^*_{\alpha\beta}, \tau^A)_p)$ and ${}^*\eta(p) = (p, (\xi_{\alpha\beta}, -\xi^*_{\alpha\beta}, \bar{\tau}^A)_p)$, $\forall p \in U \subseteq M$. The (τ^A) $(A = 3, 4, 5, 6)$ are, for our purposes, arbitrary cobivectors picked to complete the cobiframe. That is, in this construction we will only be concerned with the one-two blades, i.e., in the first and second cobivectors of the cobiframe. Any ${}^*\xi \in [{}^*\xi]$ will be called an *extremal gauge* of L^2M^*. The corresponding dual gauge of L^2M will be labeled ξ.

An example of the use of more than just two cobivectors associated with a cobiframe can be found in Norris and Davis (1979). This example considers the possibility of extending the above bivector formalism to Einstein-Yang-Mills spacetimes. Clearly, the increased number of field strengths associated with a Yang-Mills gauge theory would lead to a corresponding increase in the number of special cobivectors picked to fill out the cobiframes. Note that a generalization of the above formalism to Einstein-Yang-Mills spacetimes is a real possibility in the sense that the structure group of the biframe bundle is $Gl(6)$, which is certainly larger than most non-Abelian gauge groups associated with standard physical theories.

Consistent with the above construction, we next define a special gauge transformation. Let h: $U \to Gl(6)$, $U \subseteq M$, be defined by

$$h(p) = \begin{pmatrix} \cos\alpha(p) & \sin\alpha(p) & 0 \\ -\sin\alpha(p) & \cos\alpha(p) & 0 \\ 0 & 0 & I_4 \end{pmatrix} \tag{5.1}$$

where α: $U \to \mathbb{R}$ and the above holds for all $p \in U$. The 4×4 identity matrix I_4 could be replaced by a general $Gl(4)$ matrix. However, again we are interested here only in the first and second blades and thus we simplify.

Given ${}^*\xi$ an extremal section of L^2M^*, a new section ${}^*\Sigma = {}^*\xi \cdot h$ at a point $p \in M$ has the form ${}^*\Sigma(p) = (p, (\Sigma_{\alpha\beta}, -\Sigma^*_{\alpha\beta}, \tau^A)_p)$. Here, $\Sigma_{\alpha\beta}$ and $\Sigma^*_{\alpha\beta}$ are precisely a duality rotation of the extremal fields $\xi_{\alpha\beta}$ and $\xi^*_{\alpha\beta}$ as in (4.9) and (4.10). *Thus, duality rotations correspond to special section changes of* L^2M^*.

This model, in which the extremal fields are part of a bivector basis and duality rotations are special $Gl(6)$ transformations on this basis, helps to clarify several aspects related to the RMW problem. For example, in typical discussions concerning the RMW problem it is usually shown that expressions such as $\xi_{\alpha\beta}\xi^{\alpha\beta}$ and $\xi^*_{\alpha\beta}\xi^{\alpha\beta}$ are not duality invariant. From the above discussion, $^*\xi(p)=(p,(\xi_{\alpha\beta},-\xi^*_{\alpha\beta},\tau^A)_p)$ and thus the expressions in question are

$$\xi_{\alpha\beta}\xi^{\alpha\beta}=\tau^{(1)}_{\alpha\beta}\tau^{\alpha\beta(1)} \tag{5.2}$$

$$-\xi^*_{\alpha\beta}\xi^{\alpha\beta}=\tau^{(2)}_{\alpha\beta}\tau^{\alpha\beta(1)} \tag{5.3}$$

These correspond to components of $\tau^a_{\alpha\beta}\tau^{\alpha\beta b}$, and clearly are not invariant under $Gl(6)$ transformations (Norris and Davis, 1979).

The bitorsion structure equations (3.3) take an interesting form when pulled back in an extremal section. The first and second components $(a=1,2)$ can be recast in the form

$$\nabla_\mu\xi^{*\lambda\mu}-{}_\xi\alpha_\mu\xi^{\lambda\mu}+{}_\xi\beta_\mu\xi^{*\lambda\mu}={}^*_\xi\Theta^{\lambda(1)} \tag{5.4}$$

$$\nabla_\mu\xi^{\lambda\mu}+{}_\xi\alpha_\mu\xi^{*\lambda\mu}+{}_\xi\beta_\mu\xi^{\lambda\mu}={}^*_\xi\Theta^{\lambda(2)} \tag{5.5}$$

These equations are clearly a generalization of (4.12) and (4.13), and will be referred to as the *generalized extremal identities.* The two vectors α_μ and β_μ will be called *generalized complexion vectors.* Explicit formulas for α_μ and β_μ will be given below in special cases. Note that the bitorsion $^*\Theta^{\lambda a}$ acts as a geometrical source in the bivector field equations (5.4) and (5.5). Thus, the generalized extremal identities, which are central to the RMW problem, occur naturally in biframe bundle geometry. They are simply two components of the pullback of the bitorsion structure equations in an extremal gauge.

In the special case that ${}_\xi\alpha_\mu=\partial_\mu\alpha$ and ${}_\xi\beta_\mu=0$, a new section $^*f={}^*\xi\cdot h$ of L^2M^* can be defined. In this new gauge the generalized extremal equations (5.4) and (5.5) reduce to

$$\nabla_\alpha f^{*\lambda\alpha}={}^*_f\Theta^{\lambda(1)} \tag{5.6}$$

$$\nabla_\alpha f^{\lambda\alpha}={}^*_f\Theta^{\lambda(2)} \tag{5.7}$$

Relations (5.6) and (5.7) are the *Maxwell equations with geometrical (bitorsion) sources.* The geometrical richness of the biframe bundle in conjunction with ARMW spacetimes has thus led to a geometrization of the Maxwell equations. These field equations are a special case of the bitorsion structure equations, and the bitorsion itself plays the role of a geometrical source.

The existence of geometrical sources for the Maxwell equations on the biframe bundle leads immediately to the question of extending the RMW program to include geometrical sources. In the next section we show that such an extension can be accomplished in a partially geometrical manner.

6. AN EXTENSION OF THE RMW PROGRAM TO INCLUDE SOURCES

To consider Einstein–Maxwell spacetimes with sources, one needs not only sources for the Maxwell equations (here geometrical), but also nonelectromagnetic sources for the Einstein equations (here nongeometrical). Fortunately, nongeometrical sources for the Einstein equation can be introduced in such a manner as to keep the algebraic structure of RMW spacetimes intact. The following amounts to an extension of the algebraic structure of the Maxwell stress tensor as developed by Rainich (1925).

Let (M, g) denote a four-dimensional Riemannian spacetime geometry. The Einstein equation associated with an Einstein–Maxwell spacetime with sources can be written as

$$\tilde{G}_{\mu\nu} = T^{\text{total}}_{\mu\nu} = (f_{\alpha\mu} f^{\alpha}_{\ \nu} + f^{*}_{\alpha\mu} f^{*\alpha}_{\ \nu}) + T^{s}_{\mu\nu} \tag{6.1}$$

Here $T^{s}_{\mu\nu}$ represents the nongeometrical, nonelectromagnetic source stress tensor. For example, this part of the stress tensor for a charged fluid takes the form $T^{s}_{\alpha\beta} = \mu u_\alpha u_\beta$, where u_α is the four-velocity of the fluid and μ is the energy density [see, for example, Synge (1960)].

Note that the difference $(\tilde{G}_{\mu\nu} - T^{s}_{\mu\nu})$ has the quadratic bivector structure which is characteristic of source-free Einstein–Maxwell spacetimes. That is, given any Einstein–Maxwell spacetime with sources, we can write $(\tilde{G}_{\mu\nu} - T^{s}_{\mu\nu}) = (f_{\alpha\mu} f^{\alpha}_{\ \nu} + f^{*}_{\alpha\mu} f^{*\alpha}_{\ \nu})$. Thus, to extend the notion of ARMW spacetimes to include sources, we consider a triple (M, g, T^s), where (M, g) is a four-dimensional Riemannian spacetime and $T^{s}_{\mu\nu}$ is some given nongeometrical, nonelectromagnetic source stress tensor. Given such a triple (M, g, T^s), define the difference tensor

$$T_{\mu\nu} \overset{d}{=} \tilde{G}_{\mu\nu} - T^{s}_{\mu\nu} \tag{6.2}$$

An *algebraic Rainich spacetime* (AR) is any triple (M, g, T^s) such that the difference tensor $T_{\mu\nu}$ as in (6.2) satisfies the following conditions [cf. (4.4)–(4.6)]:

$$T = 0 \tag{6.3}$$

$$T^{\beta}_{\alpha} T^{\gamma}_{\beta} = \tfrac{1}{4} K \delta^{\gamma}_{\alpha} T_{\theta\phi} T^{\theta\phi} \tag{6.4}$$

$$T_{00} \geq 0 \tag{6.5}$$

Furthermore, the spacetime will be referred to as non-null for $K = 1$, and null for $K = 0$. As in the standard RMW formalism, the algebraic conditions (6.3)–(6.5) are necessary and sufficient to guarantee that the difference tensor $T_{\mu\nu}$ can be written in the form

$$T_{\mu\nu} = \xi_{\alpha\mu}\xi^{\alpha}_{\ \nu} + \xi^{*}_{\alpha\mu}\xi^{*\alpha}_{\ \nu}$$
$$= \Sigma_{\alpha\mu}\Sigma^{\alpha}_{\ \nu} + \Sigma^{*}_{\alpha\mu}\Sigma^{*\alpha}_{\ \nu} \tag{6.6}$$

Here, $\xi_{\mu\nu}$ is now the *extremal Maxwell square root* of $T_{\mu\nu}$, with dual $\xi^{*}_{\mu\nu}$, and the cobivectors $\Sigma_{\mu\nu}$ and $\Sigma^{*}_{\mu\nu}$ are obtained by a duality rotation [cf. (4.9) and (4.10)] of the extremal fields through a complexion angle α. Given $\xi_{\mu\nu}$ and $\xi^{*}_{\mu\nu}$, one can then define extremal sections of L^2M^* as before.

Each AR spacetime leads to an equivalence class $[\xi]$ of extremal sections of L^2M. Each such $\xi \in [\xi]$ in turn leads to the generalized extremal identities as given in (5.4) and (5.5). These can be solved for the two generalized complexion vectors α_μ and β_μ. Because of relation (6.6), these can be rewritten using bivector identities [see, for example, Misner and Wheeler (1957)] for non-null AR spacetimes as

$${}_{\xi}\alpha_{\rho} = \mathscr{E}_{\rho\lambda\mu\nu}\left\{\frac{T^{\nu}_{\sigma}\nabla^{\mu}T^{\lambda\sigma}}{T_{\theta\phi}T^{\theta\phi}}\right\} + \frac{2({}^{*}_{\xi}\Theta^{\lambda(1)}\xi_{\lambda\rho} + {}^{*}_{\xi}\Theta^{\lambda(2)}\xi^{*}_{\lambda\rho})}{T_{\theta\phi}T^{\theta\phi}} \tag{6.7}$$

$${}_{\xi}\beta_{\rho} = 4\left\{\frac{T^{\sigma}_{\rho}\nabla_{\alpha}T^{\alpha}_{\sigma}}{T_{\theta\phi}T^{\theta\phi}}\right\} + \frac{2({}^{*}_{\xi}\Theta^{\lambda(1)}\xi^{*}_{\lambda\rho} - {}^{*}_{\xi}\Theta^{\lambda(2)}\xi_{\lambda\rho})}{T_{\theta\phi}T^{\theta\phi}} \tag{6.8}$$

Clearly, (6.7) is a generalization of the form of the standard RMW complexion vector α_μ as given in (4.8). From (6.7) and (6.8), β_ρ is a new complexion vector, which is analogous to α_μ. Note that β_ρ does not appear in the standard RMW formalism. The new terms involving ${}^{*}_{\xi}\Theta^{\lambda a}$ are new geometrical source terms. The reduction of relations (6.7) and (6.8) back to the standard form of the RMW problem will be discussed below.

By an *Einstein–Maxwell spacetime with partially geometrical sources* we will mean a triple (M, g, T^s) satisfying

$$\tilde{G}_{\mu\nu} = (f_{\alpha\mu}f^{\alpha}_{\ \nu} + f^{*}_{\alpha\mu}f^{*\alpha}_{\ \nu}) + T^{s}_{\mu\nu} \tag{6.9}$$

$$\nabla_{\alpha}f^{*\lambda\alpha} = {}^{*}_{f}\Theta^{\lambda(1)} \tag{6.10}$$

$$\nabla_{\alpha}f^{\lambda\alpha} = {}^{*}_{f}\Theta^{\lambda(2)} \tag{6.11}$$

for some section $^*f = {}^*f \cdot h$ (a duality rotated section) of L^2M^*. Our discussion above shows that the RMW problem can be extended to include sources in a partially geometrical manner, and the generalization of the RMW result can be stated as follows:

An AR spacetime is a non-null Einstein–Maxwell spacetime with partially geometrical sources if and only if in an extremal gauge ${}_{\xi}\alpha_{\mu} = \partial_{\mu}\alpha$ *and* ${}_{\xi}\beta_{\mu} = 0$.

It is clear that the bitorsion acts as a geometrical source for the Maxwell equations, while it was necessary to assume a nongeometrical source for the Einstein equation.

Notice that if $T^s_{\mu\nu}=0$, then $T_{\mu\nu}=\tilde{G}_{\mu\nu}=\tilde{R}_{\mu\nu}$. Furthermore, if the bitorsion ${}^*_f\Theta^{\lambda a}=0$, then relations (6.9)–(6.11) reduce precisely to the source-free Einstein–Maxwell spacetimes [i.e., (4.1)–(4.3)]. Simultaneously, relation (6.7) reduces to the definition of the standard RMW complexion vector given in (4.8). Also, in this special case ${}_\xi\beta_\mu=0$ can be shown to be trivially satisfied through the doubly contracted Bianchi identities of the Riemannian curvature. Hence, the new complexion vectors α_μ and β_μ given in (6.7) and (6.8) are generalizations of the standard RMW complexion vector.

An interesting second special case of the above formalism is to consider the case when $T^s_{\mu\nu}=0$, but let the bitorsion ${}^*_f\Theta^{\lambda a}$ be nonzero. This would seem to imply a potential flaw in the theory, as (6.9)–(6.11) would allow sources for the Maxwell equations which did not appear as gravitational sources for the Einstein equation. However, it can be shown that $T^s_{\mu\nu}=0$ in conjunction with the condition ${}_\xi\beta_\mu=0$ [which again is necessary and sufficient for (6.9)–(6.11)] in fact forces the first and second components of the bitorsion to vanish, that is, ${}^*_f\Theta^{\lambda(1)}=0$ and ${}^*_f\Theta^{\lambda(2)}=0$. Thus, the sources obey the empirical relation that the electromagnetic sources ${}^*_f\Theta^{\lambda a}$ can vanish while the gravitational sources need not vanish, but not vice versa.

7. CONCLUSIONS

The biframe bundle is a natural geometrical arena in which to reformulate the RMW theory. Each ARMW spacetime leads to an equivalence class of extremal sections of the biframe bundle. Further, duality rotations correspond to special section changes of L^2M. In the standard RMW formalism, the Maxwell fields can be recovered from the geometry, but they play no fundamental geometrical role within the context of the given Riemannian geometry. The Maxwell fields play a more direct geometrical role in the biframe bundle, as they build part of a biframe. Furthermore, the Maxwell equations themselves are geometrized on the biframe bundle, as they are a special case of the bitorsion structure equations.

The RMW program can be extended to include Einstein–Maxwell spacetimes with partially geometrical sources. A key step in this extension is the introduction of algebraic Rainich (AR) spacetimes. Each AR spacetime is a triple (M, g, T^s) with certain algebraic properties, where (M, g) is a four-dimensional Riemannian spacetime and T^s represents all nongeometrical, nonelectromagnetic gravitational sources for the Einstein tensor. The algebraic structure associated with AR spacetimes completely parallels that of ARMW spacetimes, the key difference being that the AR spacetimes do not preclude the possibility of nonzero sources for the coupled Einstein–Maxwell equations.

The biframe bundle in conjunction with algebraic Rainich spacetimes leads to a generalization of the standard RMW program. This generalization includes geometrical sources for the Maxwell equations and nongeometrical sources for the Einstein equation. Further, the reformulation introduces two generalized complexion vectors as opposed to the single complexion vector associated with the usual RMW theory. In the special case of no sources, the entire formalism reduces to the standard RMW theory.

The standard RMW theory provides necessary and sufficient conditions on an arbitrary spacetime (M, g) in order for this spacetime to be a non-null source-free Einstein–Maxwell spacetime. The generalization of the RMW program presented here provides necessary and sufficient conditions on an arbitrary triple (M, g, T^s) in order for this triple to be a non-null Einstein–Maxwell spacetime with partially geometrical sources.

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